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Advanced Linear Algebra (MA 409)
Problem Sheet - 9

Invertibility and Isomorphisms

1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $T : V \to W$ is linear, and A and B are matrices.
   (a) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$.
   (b) T is invertible if and only if T is one-to-one and onto.
   (c) $T = L_A$, where $A = \left[T\right]_\alpha^\beta$.
   (d) $M_{2\times3}(F)$ is isomorphic to F^5.
   (e) $P_n(F)$ is isomorphic to $P_m(F)$ if and only if $n = m$.
   (f) $AB = I$ implies that A and B are invertible.
   (g) If A is invertible, then $(A^{-1})^{-1} = A$.
   (h) A is invertible if and only if L_A is invertible.
   (i) A must be square in order to possess an inverse.

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.
   (c) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$.
   (d) $T : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $\mathrm{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$.
   (e) $T : M_{2\times2}(\mathbb{R}) \to M_{2x2}(\mathbb{R})$ defined by $\mathrm{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

3. Which of the following pairs of vector spaces are isomorphic? Justify your answers.
   (a) F^3 and $P_3(F)$.
   (b) F^4 and $P_3(F)$.
   (c) $M_{2\times2}(\mathbb{R})$ and $P_3(\mathbb{R})$.
   (d) $V = \{A \in M_{2\times2}(\mathbb{R}) : tr(A) = 0\}$ and $\mathbb{R}^4$.

4. Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

6. Prove that if A is invertible and $AB = O$, then $B = O$.

7. Let A be an $n \times n$ matrix.

   (a) Suppose that $A^2 = O$. Prove that A is not invertible.

   (b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

8. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

9. Let A and B be $n \times n$ matrices such that $AB = I_n$.

   (a) Use the above exercise to conclude that A and B are invertible.

   (b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are, in effect, saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)

   (c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

10. Define

$$T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \text{ by } T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}.$$

    Show that the linear transformation T is one-to-one.

    [Hint: Lagrange interpolation formula].

11. Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.

12. Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}.$$

    Construct an isomorphism from V to F^3.

13. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

14. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

15. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

    (a) Prove that $T(V_0)$ is a subspace of W.

    (b) Prove that $\dim(V_0) = \dim(T(V_0))$.

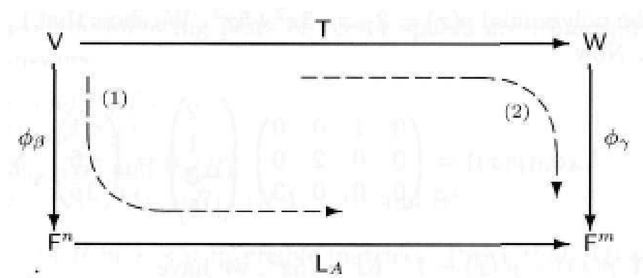
Let V and W be vector spaces of dimension n and m, and let $T : V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. Here $\phi_\beta : V \to F^n$ defined by

$$\phi_\beta(x) = [x]_\beta \quad \text{for each } \ x \in V$$

is called the **standard representation of V with respect to β.** In a similar way ϕ_γ is defined. Using ϕ_β and ϕ_γ, we have the relationship

$$L_A\phi_\beta = \phi_\gamma T$$

between the linear transformations T and $L_A : F^n \to F^m$. Heuristically, this relationship indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ, respectively, we may "identify" T with L_A.



This diagram allows us to transfer operations on abstract vector spaces to ones on F^n and F^m.

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16. Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = f'(x).$$

Let β and γ be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively, and let $\phi_\beta : P_3(\mathbb{R}) \to \mathbb{R}^4$ and $\phi_\gamma : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the corresponding standard representations of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$. If $A = [T]_\beta^\gamma$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Show that $L_A\phi_\beta(p(x)) = \phi_\gamma T(p(x))$ for $p(x) = 1 + x + 2x^2 + x^3$.

17. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $rank(T) = rank(L_A)$ and that $nullity(T) = nullity(L_A)$, where $A = [T]_\beta^\gamma$.

18. Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. Then there exist linear transformations $T_{ij} : V \to W$ such that

$$T_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

First prove that $\{T_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $\mathcal{L}(V, W)$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[T_{ij}]_\beta^\gamma = M^{ij}$. Also there exists a linear transformation $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$ such that $\Phi(T_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

19. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define $T : P_n(F) \to F^{n+1}$ by

$$T(f) = (f(c_0), f(c_1), \ldots, f(c_n)).$$

Prove that T is an isomorphism.

Hint: Use the Lagrange polynomials associated with $c_0, c_1, \ldots, c_n$.

20. Let V denote the vector space of all sequences $\{a_n\}$ in F that have only a finite number of non-zero terms a_n. We denote the sequence $\{a_n\}$ by σ such that $\sigma(n) = a_n$ for $n = 0, 1, \ldots$.

defined in Example 5 of Section 1.2, and let $W = P(F)$. Define

$$T : V \to W \text{ by } T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i,$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

21. Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{\mathrm{T}} : V/N(T) \to Z \text{ by } \overline{\mathrm{T}}(v + N(T)) = T(v)$$

for any coset $v + N(T)$ in $V/N(T)$.

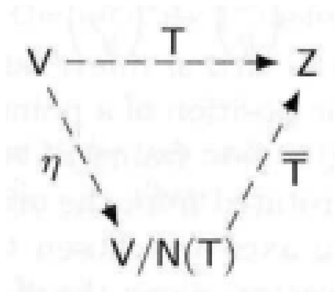
(a) Prove that $\overline{\mathrm{T}}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $T(v) = T(v')$.

(b) Prove that $\overline{\mathrm{T}}$ is linear.

(c) Prove that $\overline{\mathrm{T}}$ is an isomorphism.

(d) Prove that the diagram shown in the figure commutes; that is, prove that $T = \overline{\mathrm{T}}_\eta$.

22. Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. Let $C(S, F)$ denote the vector space of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but



a finite number of vectors in S. Let $\Psi : C(S, F) \to V$ be defined by $\Psi(f) = 0$ if f is the zero function, and

$$\Psi(f) = \sum_{s \in S, f(s) \neq 0} f(s)s,$$

otherwise. Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

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